

MATHEMATICS

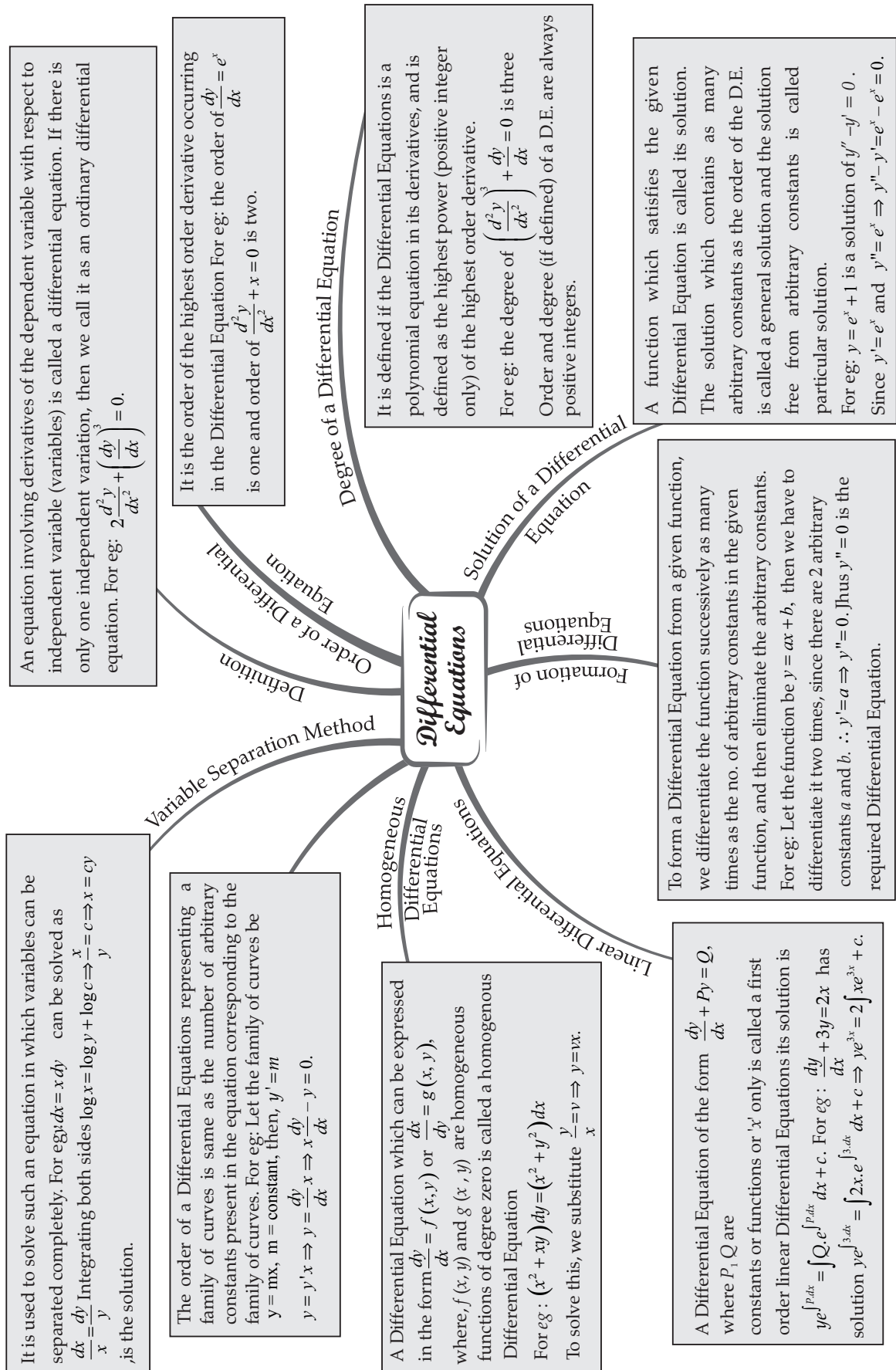


Differential equations

- 1. Differential Equation:** An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.
- 2. Linear and non-linear differential equation:** A differential equation is said to be linear if unknown function (dependent variable) as its derivative which occurs in the equation, occur only in the first degree, and are not multiplied together. Otherwise, the differential equation is said to be non-linear.
- 3. Order:** Order of a differential equation is the order of the highest order derivative occurring in the differential equation.
- 4. Degree:** Degree of a differential equation is defined if it is a polynomial equation in its derivatives.
- 5.** Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
- 6. Solution:** A function which satisfies the given differential equation is called its solution.
- 7. General Solution:** The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution.
- 8. Particular Solution:** The solution free from arbitrary constants is called particular solution.
- 9.** To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.
- 10. Variable Separable method:** Variable separable method is used to solve such an equation in which variables can be separated completely i.e., terms containing y should remain with dy and terms containing x should remain with dx .
- 11.** A differential equation which can be expressed in the form $\frac{dy}{dx} f(x, y)$ or $\frac{dx}{dy} g(x, y)$ where, $f(x, y)$ and $g(x, y)$ are homogenous functions of degree zero is called a homogeneous differential equation.
- 12.** A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only is called a first order linear differential equation.

MIND MAP : LEARNING MADE SIMPLE

CHAPTER - 9



Important Questions

Multiple Choice questions-

1. The degree of the differential equation:

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0 \text{ is}$$

- (a) 3
- (b) 2
- (c) 1
- (d) not defined.

2. The order of the differential equation:

$$2x^2 \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + y = 0 \text{ is}$$

- (a) 2
- (b) 1
- (c) 0
- (d) not defined.

3. The number of arbitrary constants in the general solution of a differential equation of fourth order is:

- (a) 0
- (b) 2
- (c) 3
- (d) 4.

4. The number of arbitrary constants in the particular solution of a differential equation of third order is:

- (a) 3
- (b) 2

(c) 1

(d) 0.

5. Which of the following differential equations has $y = c_1 e^x + c_2 e^{-x}$ as the general solution?

(a) $\frac{d^2y}{dx^2} + y = 0$

(b) $\frac{d^2y}{dx^2} - y = 0$

(c) $\frac{d^2y}{dx^2} + 1 = 0$

(d) $\frac{d^2y}{dx^2} - 1 = 0$

6. Which of the following differential equations has $y = x$ as one of its particular solutions?

(a) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = x$

(b) $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = x$

(c) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$

(d) $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = 0$

7. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(a) $e^x + e^{-y} = c$

(b) $e^x + e^y = c$

(c) $e^{-x} + e^y = c$

(d) $e^{-x} + e^{-y} = c.$

8. Which of the following differential equations cannot be solved, using variable separable method?

(a) $\frac{dy}{dx} + e^{x+y} + e^{-x+y}$

(b) $(y^2 - 2xy) dx = (x^2 - 2xy) dy$

(c) $xy \frac{dy}{dx} = 1 + x + y + xy$

(d) $\frac{dy}{dx} + y = 2.$

9. A homogeneous differential equation of the form $\frac{dy}{dx} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution.

(a) $y = vx$

(b) $v = yx$

(c) $x = vy$

(d) $x = v$

10. Which of the following is a homogeneous differential equation?

(a) $(4x + 6y + 5)dy - (3y + 2x + 4)dx = 0$

(b) $xy dx - (x^3 + y^2)dy = Q$

(c) $(x^3 + 2y^2) dx + 2xy dy = 0$

(d) $y^2 dx + (x^2 - xy - y^2)dy = 0.$

Very Short Questions:

1. Find the order and the degree of the differential equation: $x^2 \frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^4$ (Delhi 2019)

2. Determine the order and the degree of the differential equation: $\left(\frac{dy}{dx}\right)^3 + 2y \frac{d^2y}{dx^2} = 0$ (C.B.S.E. 2019 C)

3. Form the differential equation representing the family of curves: $y = b(x + a)$, where « and b are arbitrary constants. (C.B.S.E. 2019 C)

4. Write the general solution of differential equation:

$$\frac{dy}{dx} = e^{x+y} \text{ (C.B.S.E. Sample Paper 2019-20)}$$

5. Find the integrating factor of the differential equation:

$$y \frac{dy}{dx} - 2x = y^3 e^{-y}$$

6. Form the differential equation representing the family of curves $y = a \sin(3x - b)$, where a

and b are arbitrary constants. (C.B.S.E. 2019C)

Short Questions:

1. Determine the order and the degree of the differential equation:
2. Form the differential equation representing the family of curves: $y = e^{2x} (a + bx)$, where 'a' and 'h' are arbitrary constants. (Delhi 2019)
3. Solve the following differentia equation:

$$\frac{dy}{dx} + y = \cos x - \sin x \text{ (Outside Delhi 2019)}$$

4. Solve the following differential equation:

$$\frac{dx}{dy} + x = (\tan y + \sec 2y). \text{ (Outside Delhi 2019 C)}$$

Long Questions:

1. Find the area enclosed by the circle:

$$x^2 + y^2 = a^2. \text{ (N.C.E.R.T.)}$$

2. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. (C.B.S.E. 2018)
3. Find the area bounded by the curves $y = \sqrt{x}$, $2y + 3 = Y$ and Y-axis. (C.B.S.E. Sample Paper 2018-19)
4. Find the area of region:

$$\{(x,y): x^2 + y^2 < 8, x^2 < 2y\}. \text{ (C.B.S.E. Sample Paper 2018-19)}$$

Case Study Questions:

1. If the equation is of the form $\frac{dy}{dx} + py = Q$, where P, Q are functions of x, then the solution of the differential equation is given by $y e^{\int p dx} = \int Q e^{\int p dx} dx + c$, where $e^{\int p dx}$ is called the integrating factor (I.F.).

Based on the above information, answer the following questions.

i. The integrating factor of the differential equation

$$\sin x \frac{dy}{dx} + 2y \cos x = 1 \text{ is } (\sin x)^\lambda, \text{ where } \lambda =$$

- a. 0
- b. 1
- c. 2
- d. 3

ii. Integrating factor of the differential equation $(1 - x^2) \frac{dy}{dx} - xy = 1$ is:

- a. $-x$
- b. $\frac{x}{1+x^2}$
- c. $\sqrt{1 - x^2}$
- d. $\frac{1}{2} \log(1 - x^2)$

iii. The solution of $\frac{dy}{dx} + y = e^{-x}$, $y(0) = 0$, is:

- a. $y = e^x(x - 1)$
- b. $y = xe^{-x}$
- c. $y = xe^{-x} + 1$
- d. $y = (x + 1)e^{-x}$

iv. General solution of $\frac{dy}{dx} + y \tan x = \sec x$ is:

- a. $y \sec y = \tan x + c$
- b. $y \tan x = \sec x + c$
- c. $\tan x = y \tan x + c$
- d. $x \sec x = \tan y + c$

v. The integrating factor of differential equation $\frac{dy}{dx} - 3y = \sin 2x$ is:

- a. e^{3x}
- b. e^{-2x}
- c. e^{-3x}
- d. xe^{-3x}

2. If the equation is of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ or $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, where $f(x, y)$, $g(x, y)$ are homogeneous functions of the same degree in x and y , then put $y = vx$ And

$\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$, so that the dependent variable y is changed to another variable v and then apply variable separable method.

Based on the above information, answer the following questions.

i. The general solution of $x^2 \frac{dy}{dx} = x^2 + xy + y^2$ is:

- a. $\tan^{-1} \frac{x}{y} = \log |x| + c$
- b. $\tan^{-1} \frac{y}{x} = \log |x| + c$
- c. $y = x \log |x| + c$
- d. $x = y \log |y| + c$

ii. Solution of the differential equation $2xy \frac{dy}{dx} = x^2 + 3y^2$ is:

- a. $x^3 + y^2 = cx^2$
- b. $\frac{x^2}{2} + \frac{y^3}{3} = y^2 + c$
- c. $x^2 + y^3 = cx^2$
- d. $x^2 + y^2 = cx^3$

iii. General solution of the differential equation $(x^2 + 3xy + y^2) dx - x^2 dy = 0$ is:

a. $\frac{x+y}{y} - \log x = c$

b. $\frac{x+y}{y} + \log x = c$

c. $\frac{x}{x+y} - \log x = c$

d. $\frac{x}{x+y} + \log x = c$

iv. General solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} \left\{ \log \left(\frac{y}{x} \right) + 1 \right\}$ is:

a. $\log(xy) = c$

b. $\log y = cx$

c. $\log \frac{y}{x} = cx$

d. $\log x = cy$

v. Solution of the differential equation $\left(x \frac{dy}{dx} - y \right) e^{\frac{y}{x}} = x^2 \cos x$ is:

a. $e^{\frac{y}{x}} - \sin x = c$

b. $e^{\frac{y}{x}} + \sin x = c$

c. $e^{\frac{-y}{x}} - \sin x = c$

d. $e^{\frac{-y}{x}} + \sin x = c$

Answer Key-

Multiple Choice questions-

1. Answer: (a) 3

2. Answer: (a) 2

3. Answer: (d) 4.

4. Answer: (d) 0.
5. Answer: (b) $\frac{d^2y}{dx^2} - y = 0$
6. Answer: (c) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$
7. Answer: (a) $e^x + e^{-y} = c$
8. Answer: (b) $(y^2 - 2xy) dx = (x^2 - 2xy) dy$
9. Answer: (c) $x = vy$
10. Answer: (d) $y^2 dx + (x^2 - xy - y^2)dy = 0$.

Very Short Answer:

1. Solution: Here, order = 2 and degree = 1.
2. Solution: Order = 2 and Degree = 1.
3. Solution:

We have: $y = b(x + a) \dots(1)$

Diff. w.r.t. x, b.

Again diff. w.r.t. x, $\frac{d^2y}{dx^2} = 0$,

which is the reqd. differential equation.

4. Solution:

We have: $\frac{dy}{dx} = e^{x+y}$

$\Rightarrow e^{-y} dy = e^x dx$ [Variables Separable]

Integrating, $\int e^{-y} dy + c = \int e^x dx$

$\Rightarrow -e^{-y} + c = e^x$

$\Rightarrow e^x + e^{-y} = c$.

5. Solution:

The given equation can be written as.

$$\frac{dy}{dx} - \frac{2x}{y} = y^2 e^{-y}.$$

$$\therefore \text{I.F.} = e^{-\int \frac{2}{y} dy}$$

$$= e^{-2 \log|y|} = e^{\log \frac{1}{y^2}} = \frac{1}{y^2}$$

6. Solution:

We have: $y = a \sin(3x - b) \dots(1)$

Diff. W.r.t $y \frac{dy}{dx} = a \cos(3x - b) \dots(2)$

$$= 3a \cos(3x - b)$$

$$\frac{d^2y}{dx^2} = -3a \sin(3x - b) \dots(3)$$

$$= -9a \sin(3x - b)$$

$$= -9y \text{ [Using (1)]}$$

$$\frac{d^2y}{dx^2} + 9y = 0, m$$

which in the reqd. differential equation.

Short Answer:

1. Solution: Order = 2 and Degree = 1.

2. Solution:

We have: $y = e^{2x} (a + bx) \dots(1)$

Diff. w.r.t. $x, \frac{dy}{dx} = e^{2x} (b) + 2e^{2x} (a + bx)$

$$\Rightarrow \frac{dy}{dx} = be^{2x} + 2y \dots\dots\dots (2)$$

Again diff. w.r.t. $x,$

$$\frac{d^2y}{dx^2} = 2be^{2x} + 2^{2x}$$

$$\frac{d^2y}{dx^2} = 2 \left(\frac{dy}{dx} - 2y \right) + \frac{dy}{dx}$$

[Using (2)]

Hence, $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$, which is the reqd. differential equation.

3. Solution:

The given differential equation is:

$$\frac{dy}{dx} + y = \cos x - \sin x \quad \text{Linear Equation}$$

$$\therefore \text{I.F.} = e^{\int 1 dx} = e^x$$

The solution is :

$$y \cdot e^x = \int (\cos x - \sin x) e^x dx + C$$

$$\Rightarrow y \cdot e^x = e^x \cos x + C$$

$$\text{or } y = \cos x + C e^{-x}$$

4. Solution:

The given differential equation is:

$$\frac{dx}{dy} + x = (\tan y + \sec^2 y)$$

Linear Equation

$$\therefore \text{I.F.} = \int 1 dy = e^y$$

\therefore The solution is:

$$x \cdot e^y = \int e^y (\tan y + \sec^2 y) dy + c$$

$$\Rightarrow x \cdot e^y = e^y \tan y + c$$

$$= x = \tan y + c e^{-y}, \text{ which is the reqd. solution.}$$

Long Answer:

1. Solution:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\frac{y^2}{x^2} - 1}{\frac{2y}{x}}$$

Put $\frac{y}{x} = v$

$$\Rightarrow y = vx \text{ and so } \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore v + x \frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow \frac{x dv}{dx} = -\frac{(1+v^2)}{2v}$$

$$\Rightarrow \int \frac{dx}{x} = -\int \frac{2v dv}{1+v^2}$$

$$\log x = -\log(1+v^2) + \log C$$

$$x(1+v^2) = C$$

$$x \left(1 + \frac{y^2}{x^2} \right) = C$$

$$x^2 + y^2 = C.$$

2. Solution:

$$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{1}{(1+x^2)^2}$$

$$\text{I.F.} = e^{\int \frac{2x}{1+x^2} + e^{\log(1+x^2)}} = (1+x^2).$$

$$\text{Solution is } y(1+x^2) = \int \frac{1}{1+x^2} dx$$

$$= \tan^{-1} x + C$$

When $y = 0, x = 1,$

$$\text{then } 0 = \frac{\pi}{4} + C$$

$$C = -\frac{\pi}{4}$$

$$\therefore y(1+x^2) = \tan^{-1} x - \frac{\pi}{4}$$

$$\text{i.e, } y = \frac{\tan^{-1} x}{1+x^2} - \frac{\pi}{4(1+x^2)}$$

3. Solution:

$$\text{We have: } y = ae^{bx+5} + 5 \dots(1)$$

$$\text{Diff. w.r.t. } x, \frac{dy}{dx} = ae^{bx+5} \cdot (b) \dots(b)$$

$$\frac{dy}{dx} = dy \dots\dots(2) \text{ [Using (1)]}$$

Again diff. w.r.t x.,

$$\frac{d^2y}{dx^2} = b \frac{dy}{dx} \dots\dots\dots(3)$$

Dividing (3) by (2),

$$\frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}} = \frac{dy}{y}$$

$$\Rightarrow y \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0,$$

which is the required differential equation.

4. Solution:

The given differential equation is:

$$x dx - ye^y \sqrt{1+x^2} dy = 0$$

$$\Rightarrow \frac{x dx}{\sqrt{1+x^2}} - ye^y dy = 0 \text{ [Variables Separable]}$$

$$\text{Integrating, } \int \frac{x dx}{\sqrt{1+x^2}} - \int ye^y dy = c \quad \dots(1)$$

$$\text{Now, } \int \frac{x}{\sqrt{1+x^2}} dx = \frac{1}{2} \int (1+x^2)^{-1/2} (2x) dx$$

$$= \frac{1}{2} \frac{(1+x^2)^{1/2}}{1/2} = \sqrt{1+x^2} .$$

$$\text{And, } \int y e^y dy = y \cdot e^y - \int (1) e^y dy$$

[Integrating by parts]

$$= ye^y - e^y .$$

$$\therefore \text{From (1), } \sqrt{1+x^2} - (ye^y - e^y) = c$$

$$\Rightarrow \sqrt{1+x^2} = c + e^y (y - 1) \quad \dots(2)$$

When $x = 0, y = 1, \therefore 1 = c + c(0) \Rightarrow c = 1.$

Putting in (2), $\sqrt{1+x^2} = 1 + e^y (y - 1),$

which is the reqd. particular solution.

Case Study Answers:

1. Answer :

i. (c) 2

Solution:

The given differential equation can be written as $\frac{dy}{dx} + 2y \cot x = \operatorname{cosec} x$

$$\therefore \text{I.F} = e^{\int 2 \cot x dx} = e^{2 \log |\sin x|} = (\sin x)^2$$

$$\therefore \lambda = 2$$

ii. (c) $\sqrt{1-x^2}$

Solution:

We have, $(1-x^2) \frac{dy}{dx} - xy = 1$

$$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2} \cdot y = \frac{1}{1-x^2}$$

$$\therefore \text{I.F.} = e^{-\int \frac{x}{1-x^2} dx} = e^{\frac{1}{2} \int \frac{-2x}{1-x^2} dx}$$

$$= e^{\frac{1}{2} \log(1-x^2)} = e^{\log(1-x^2)^{\frac{1}{2}}} = \sqrt{1-x^2}$$

iii. (b) $y = xe^{-x}$

Solution:

We have, $\frac{dy}{dx} + y = e^{-x}$

It is a linear differential equation with $\text{I.F.} = e^{\int dx} = e^x$

Now, solution is $y \cdot e^x = \int e^{-x} dx + c$

$$\Rightarrow ye^x = \int dx + c$$

$$\Rightarrow ye^x = x + c$$

$$\Rightarrow y = xe^{-x} + ce^{-x}$$

$$\because y(0) = 0 \Rightarrow c = 0$$

$$\therefore y = xe^{-x}$$

iv. (a) $y \sec y = \tan x + c$

Solution:

We have, $\frac{dy}{dx} + y \tan x = \sec x$

It is a linear differential equation with,

I.F. = $e^{\int \tan x dx} = e^{\log |\sec x|} = \sec x$

Now, solution is $y \sec x = \int \sec^2 x dx + c$

$\Rightarrow y \sec x = \tan x + c$

v. (c) e^{-3x}

Solution:

We have, $\frac{dy}{dx} 3y = \sin 2x$

It is a linear differential equation with,

I.F. = $e^{\int -3dx} = e^{-3x}$

2. Answer :

i. (b) $\tan^{-1} \frac{y}{x} = \log |x| + c$

Solution:

We have, $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$

Put $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$\therefore v + x \frac{dv}{dx} = \frac{x^2 + x \times vx + v^2 x^2}{x^2} = 1 + v + v^2$

$\Rightarrow x \frac{dv}{dx} = 1 + v^2 \Rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x} + c$

$\Rightarrow \tan^{-1} v = \log |x| + c$

$\Rightarrow \tan^{-1} \frac{y}{x} = \log |x| + c$

ii. (d) $x^2 + y^2 = cx^3$

Solution:

We have,

$$2xy \frac{dy}{dx} = x^2 + 3y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$

Put $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = \frac{x^2 + 3v^2x^2}{2vx^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + 3v^2}{2v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$\Rightarrow \int \frac{2v}{1 + v^2} dv = \int \frac{dx}{x} + \log c$$

$$\Rightarrow \log |1 + v^2| = \log |x| + \log |c|$$

$$\Rightarrow \log |v^2 + 1| = \log |xc|$$

$$\Rightarrow v^2 + 1 = xc \Rightarrow \frac{y^2}{x^2} + 1 = xc$$

$$\Rightarrow x^2 + y^2 = x^3c$$

$$\text{iii. (d) } \frac{x}{x+y} + \log x = c$$

Solution:

We have,

$$(x^2 + 3xy + y^2) dx - x^2 dy = 0$$

$$\Rightarrow \frac{x^2 + 3xy + y^2}{x^2} = \frac{dy}{dx}$$

$$\text{Put } y = vx \text{ and } \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore \frac{x^2 + 3x^2v + x^2v^2}{x^2} = \left(v + x \frac{dv}{dx} \right)$$

$$\Rightarrow 1 + 3v + v^2 = v + x \frac{dv}{dx}$$

$$\Rightarrow 1 + 2v + v^2 = x \frac{dv}{dx}$$

$$\Rightarrow \int \frac{dx}{x} - \int (v+1)^{-2} = dv = c$$

$$\log x + \frac{1}{v+1} = c$$

$$\Rightarrow \log x + \frac{x}{x+y} = c$$

iv. (c) $\log \frac{y}{x} = cx$

Solution:

We have, $\frac{dy}{dx} = \frac{y}{x} \left\{ \log \left(\frac{y}{x} \right) + 1 \right\}$

Put $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$\therefore v + x \frac{dv}{dx} = v \{ \log(v + 1) \}$

$\Rightarrow x \frac{dv}{dx} = v \log v$

$\Rightarrow \int \frac{dv}{v \log v} = \int \frac{dx}{x} \Rightarrow \log | \log v | = \log |x| + \log |c|$

$\Rightarrow \log \left(\frac{y}{x} \right) = cx$

v. (a) $e^{\frac{y}{x}} - \sin x = c$

Solution:

We have, $\left(x \frac{dy}{dx} - y \right) e^{\frac{y}{x}} = x^2 \cos x$

$\Rightarrow \left(\frac{dy}{dx} - \frac{y}{x} \right) e^{\frac{y}{x}} = x \cos x$

Put $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$\Rightarrow \left(v + x \frac{dv}{dx} - v \right) e^v = x \cos x$

$\Rightarrow x e^v \frac{dv}{dx} = x \cos x$

$\Rightarrow \int e^v dv = \int \cos x dx$

$\Rightarrow e^v = \sin x + c$

$\Rightarrow e^{\frac{y}{x}} - \sin x = c$